Math 1552

Sections 6.1 and 6.2: Volumes of Revolution

Math 1552 lecture slides adapted from the course materials
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Quiz 5 is on Thursday July 22, 2021
(during the last 25 minutes of the studio session).

Topics List:

- power series
- radius of convergence and IC of power series
- Taylor polynomials
- Taylor series
- Taylor series and remainder terms + error bounds in approximating series (see lecture notes)

Learning Goals

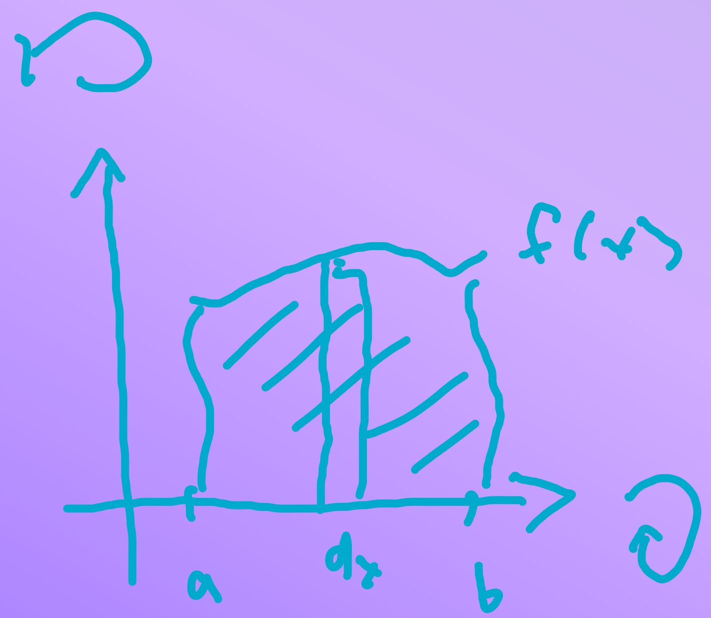
- Set up and evaluate integrals using the disk method
- Set up and evaluate integrals using the method of cylindrical shells
- Apply the “washer” method to either method above
- Adjust the standard formulas to rotate a region around any horizontal or vertical line

Volumes by the Disk Method

We can find the volume of the solid generated by revolving the region bounded by *y*=*f*(*x*), *x*=*a*, *x*=*b*, and the *x*-axis using the basic formulas:

$$V = \pi \int_a^b [f(x)]^2 \, dx \quad \text{(revolved about } x\text{-axis)}$$

$$V = \pi \int_a^b [g(y)]^2 \, dy \quad \text{(revolved about } y\text{-axis)}$$



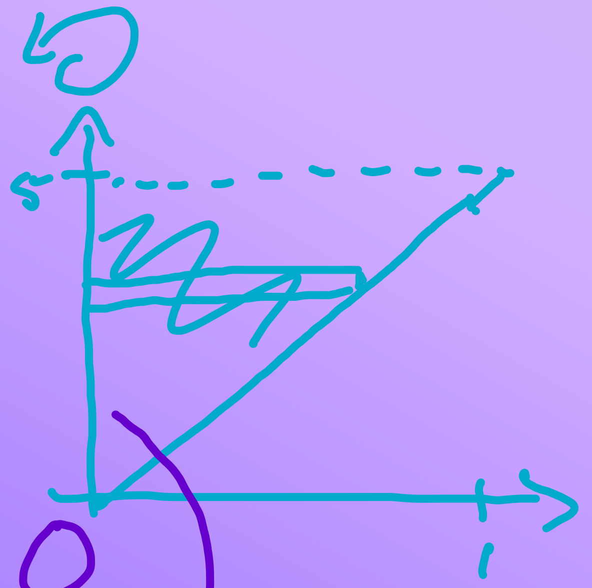
↳ example 1

Example 1:

Find the volume of the solid generated by revolving the region bounded by *y*=5*x*, *x*=0, and *y*=5 about the *y*-axis.

$\longleftrightarrow x = g(y) = \frac{y}{5}$

$\rightarrow g(y)^2$



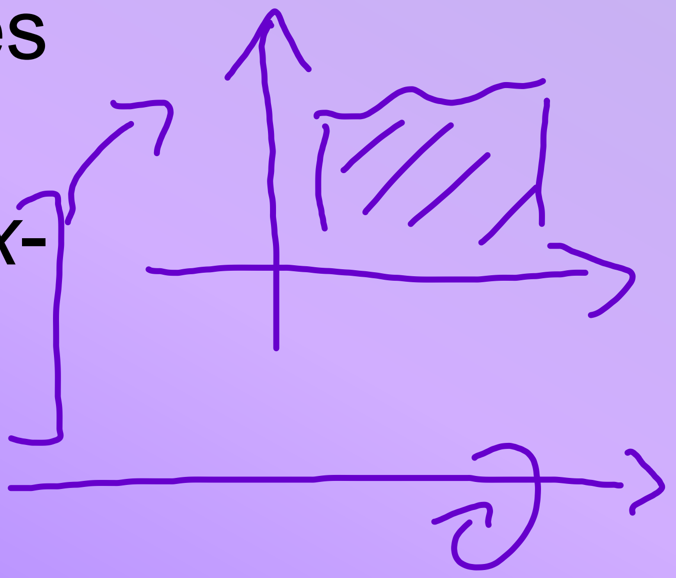
$$V = \pi \int_0^5 \frac{y^2}{25}\,dy$$

$$= \frac{\pi}{25} \frac{y^3}{3} \Big|_0^5 = \frac{\pi}{25}\left(\frac{5^3 - 0}{3}\right)$$

$$= \frac{5\pi}{3}$$

Important Notes about Disks:

- The variable of integration *always* matches the axis of revolution.
- If you revolve about a line other than the *x*- or *y*-axis, you will need to adjust the formula to find the new radius.
- If you revolve a region bounded by two curves, you will need to apply the *washer method*.



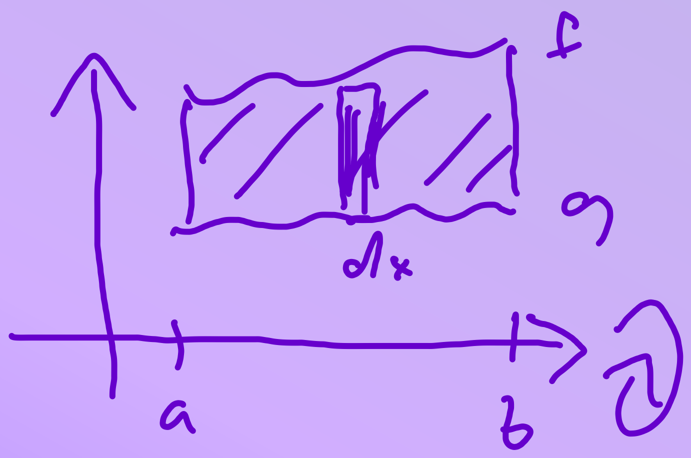
The Washer Method

When we revolve a region bounded between two curves, we have an inner and outer radius, and the volume equation is modified to:

$$V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx = \pi \int_a^b \left[(top)^2 - (bottom)^2 \right] dx$$

OR

$$V = \pi \int_a^b \left[(f(y))^2 - (g(y))^2 \right] dy = \pi \int_a^b \left[(right)^2 - (left)^2 \right] dy$$

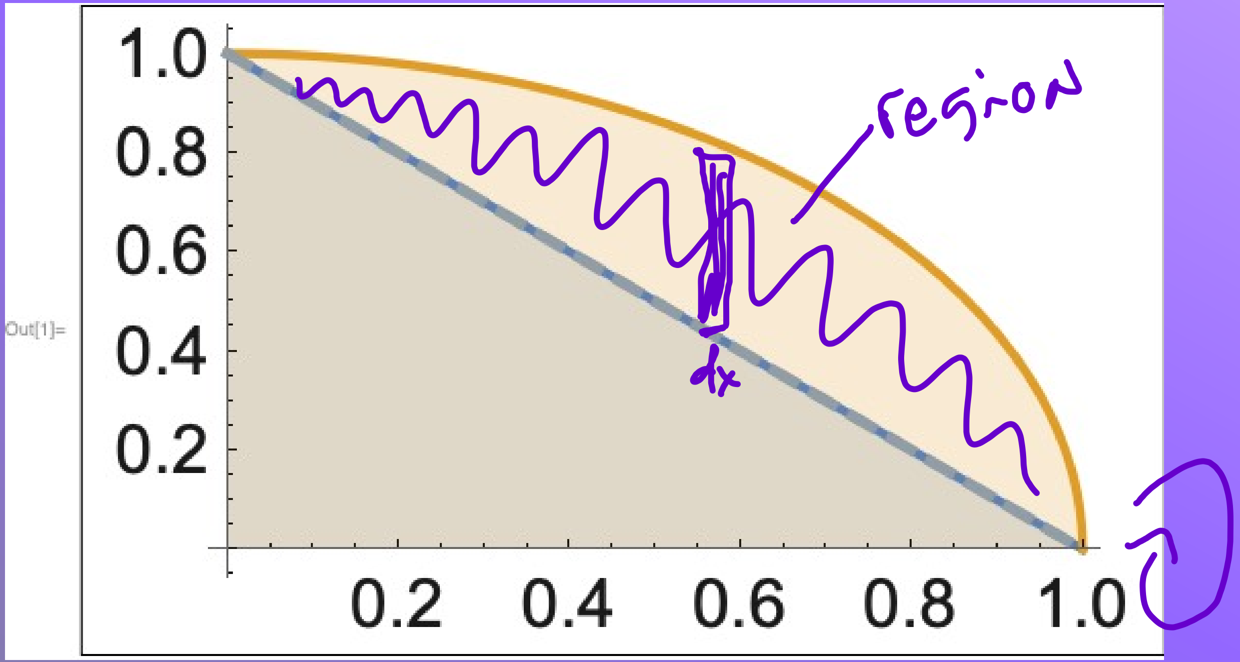


Example 2:

Find the volume of the solid generated by revolving the region bounded by

$$y_1 = \sqrt{1-x^2} \text{ and } x+y=1 \longleftrightarrow y_2 = 1-x$$

about the x-axis.



$y_1 = \sqrt{1-x^2}$

(in orange)

$y_2 = 1-x$

(in blue)

for all $x \in [0,1]$:

$y_1(x) \geqslant y_2(x)$

$$V = \pi \int_0^1 \left[y_1(x)^2 - y_2(x)^2 \right] dx$$

$$= \pi \int_0^1 \left[1 - x^2 - (1-x)^2 \right] dx$$

$$= \pi \int_0^1 \left[2x - 2x^2 \right] dx$$

$$= 2\pi \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{3} - 0 \right)$$

$$= \frac{2\pi}{6} = \frac{\pi}{3}$$

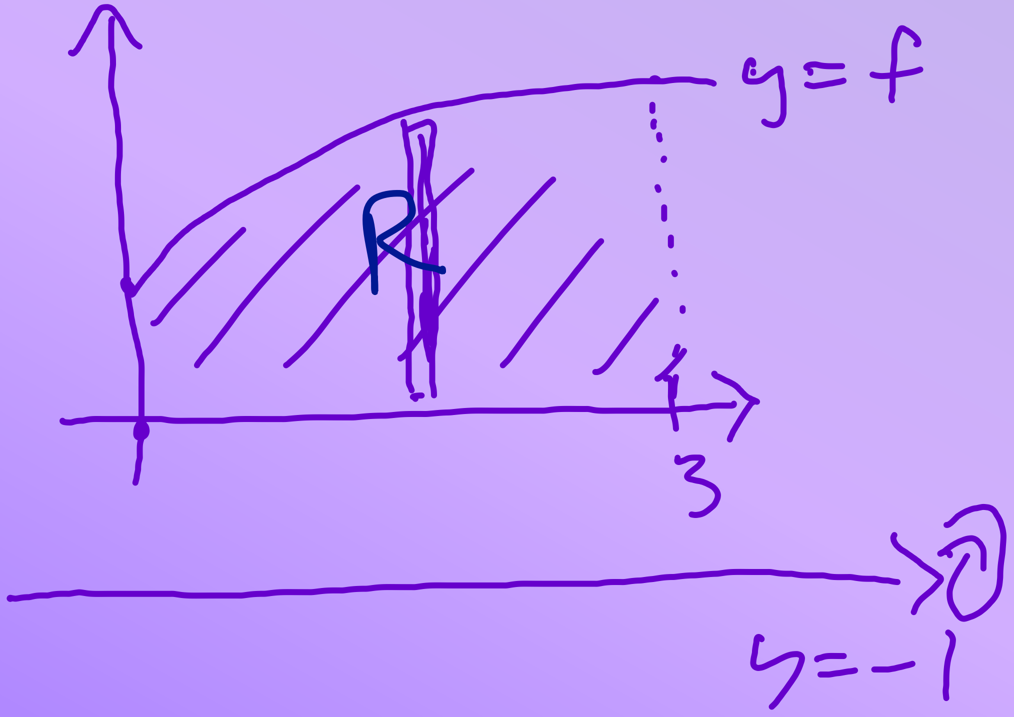
Example 3:

Find the volume of the solid generated by revolving the region bounded by:

$x=0$

$$y=\sqrt{x+1}, x=3, \text{ and the } x\text{-axis}$$

$$\text{about the line } y=-1.$$



R = shaded region

Picking up with corrections to the method for this problem explained in the lecture today:

→ the region (R) is the region bdd between the curves

$$f(x) = \sqrt{x+1}$$

$$g(x) = 0 \qquad \text{(the x-axis)}$$

for $0 \leq x \leq 3$

→ we use a variant of the washer method formula to compute the volume (V) of the solid

generated by revolving R about the line $y=-1$

→ Notice that for all $x \in [0,3]$, $f(x) \geq g(x)$

So that the top curve is f and the bottom curve is g.

→ $$V = \pi \int_0^3 \Big[(\text{distance between } f(x) \text{ and } -1)^2 - (\text{distance between } g(x) \text{ and } -1)^2 \Big] dx$$

$$= \pi \int_0^3 \Big[(f(x) - (-1))^2 - (g(x) - (-1))^2 \Big] dx$$

$$= \pi \int_0^3 \left[\left(\sqrt{x+1} + 1 \right)^2 - 1 \right] dx$$

$$= \pi \int_0^3 \left[x + 2\sqrt{x+1} \right] dx$$

$$= \pi \left(\frac{x^2}{2} + \frac{4}{3}(x+1)^{3/2} \right) \Bigg|_0^3$$

$$= \pi \left(\left(\frac{9}{2} + \frac{4}{3} \cdot \cancel{4^{3/2}} \xrightarrow{} 8 \right) - \left(0 + \frac{4}{3} \right) \right)$$

$$= \frac{83\pi}{6}$$

__Example__: Set up the integral to find the volume bounded by

$y = x + 2$ and $y = x^2$, $x \geq 0$,

about the x-axis.

(A) $V = \pi \int_{-1}^{2} \left[(x+2)^2 - (x^2)^2 \right] dx$

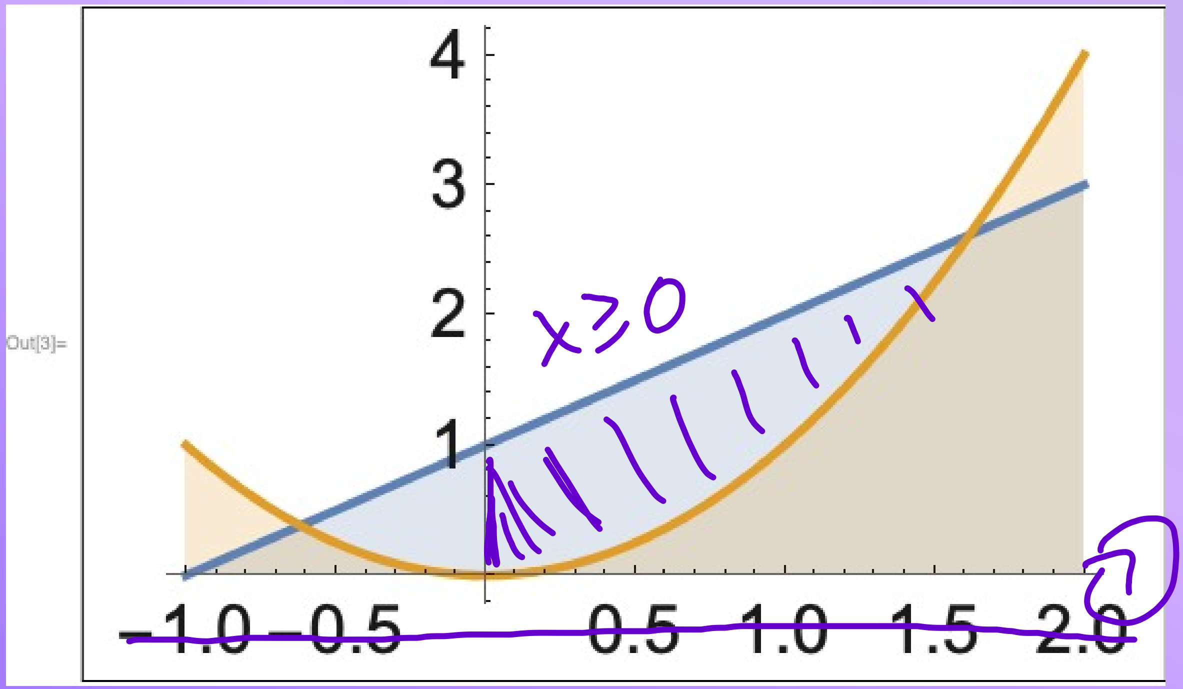
$\boxed{\text{(B)}}$ $V = \pi \int_{0}^{2} \left[(x+2)^2 - (x^2)^2 \right] dx$

(C) $V = \pi \int_{-1}^{2} \left[(x^2)^2 - (x+2)^2 \right] dx$

(D) $V = \pi \int_{0}^{2} \left[(x^2)^2 - (x+2)^2 \right] dx$

$y_1 = x + 2$ (in blue)

$y_2 = x^2$ (in orange)



for $-1 \leq x \leq 2$,

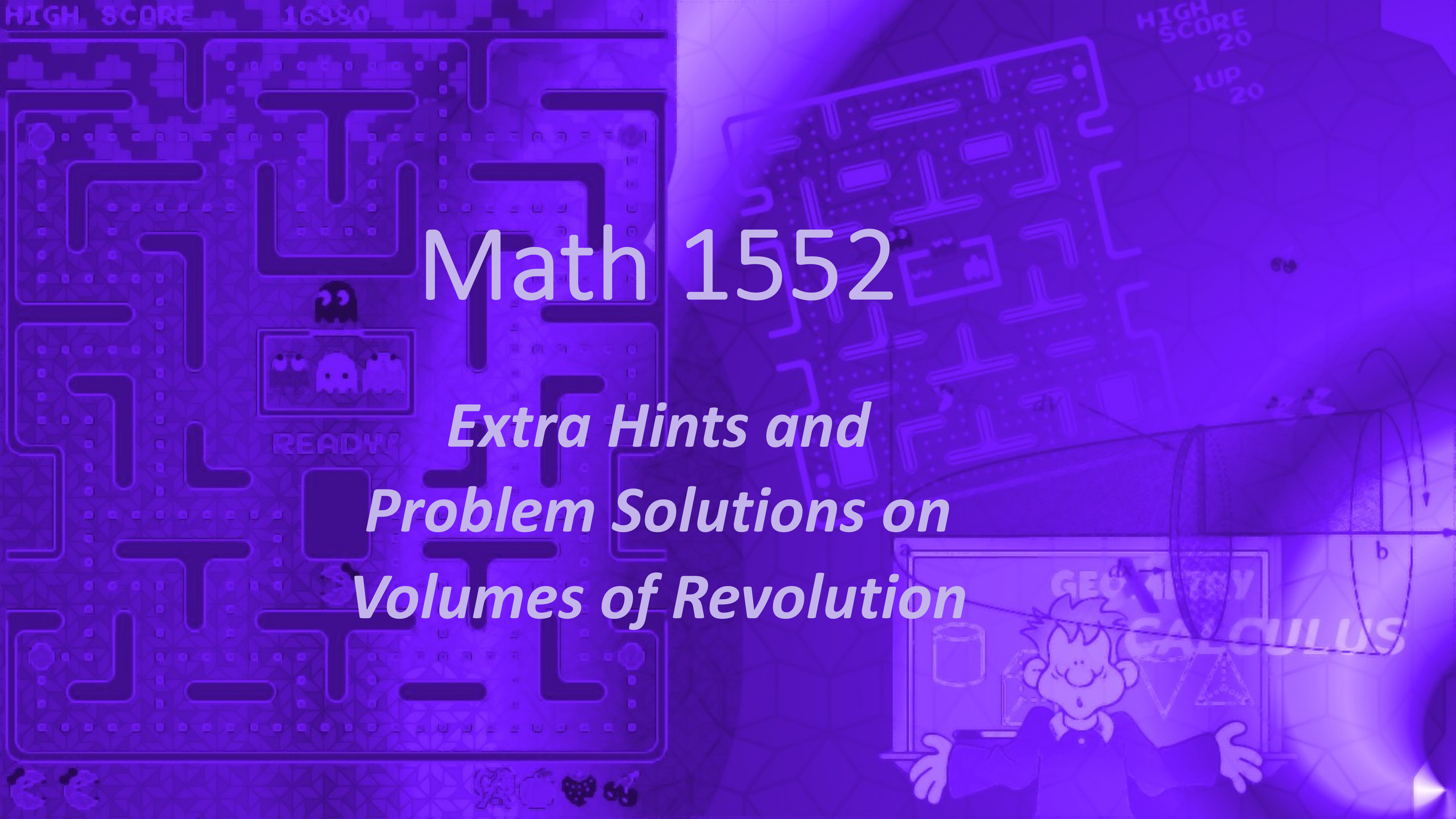
$y_1(x) \geq y_2(x)$

$\longrightarrow$ find the intersection points: $y_1 = y_2$

$$x^2 - x - 2 = 0$$

$$\Longleftrightarrow (x-2)(x+1) = 0$$

$$\Longleftrightarrow x = 2, -1$$



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Extra Hints and Problem Solutions on Volumes of Revolution

Example B:

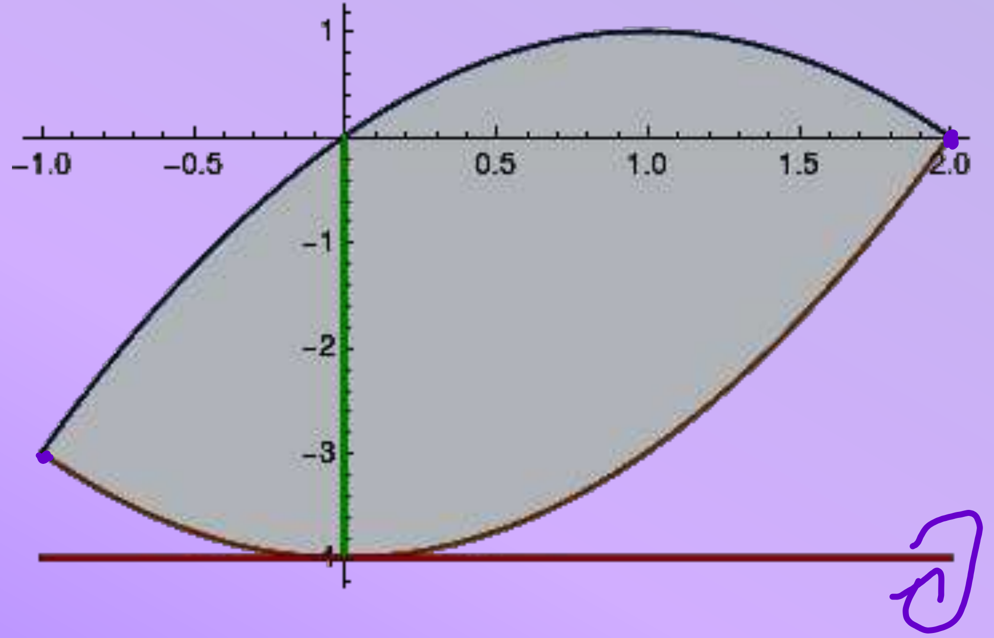
Find the volume of the solid generated by revolving the region bounded by the curves

$$y_1(x) = x^2 - 4$$ (*in orange*)

AND

$$y_2(x) = 2x - x^2$$ (*in blue*)

around the line y=-4.



Use the WASHER METHOD

(be careful to add +4 to each of the functions before squaring):

$$V = \cancel{2}\pi \times \int_a^b (\textbf{radius of washer at x})^2 \, dx$$

$$= 45\pi$$

→ find the intersection points : $y_1 = y_2$

$2x^2 - 2x - 4 = 0 \Longleftrightarrow x^2 - x - 2 = 0$

$\Longleftrightarrow (x-2)(x+1) = 0 \Longleftrightarrow x = 2, -1$

→ for all $-1 \leq x \leq 2$, $y_2(x) \geq y_1(x)$

→ formula for the volume is a variant of the washer method:

$$V = \pi \int_{-1}^{2} \left[(\text{distance between top curve at } x \text{ and } y=-4)^2 - (\text{distance between bottom curve at } x \text{ and } y=-4)^2 \right] dx$$

$$= \pi \int_{-1}^{2} \left[\left(y_2(x) - (-4) \right)^2 - \left(y_1(x) + 4 \right)^2 \right] dx$$

$$= \pi \int_{-1}^{2} \left[\left(2x - x^2 + 4 \right)^2 - \left(x^2 \right)^2 \right] dx$$

$$= \pi \int_{-1}^{2} \left[16 + 16x - 4x^2 - 4x^3 \right] dx$$

$$= \pi \left(16x + 8x^2 - \frac{4x^3}{3} - x^4 \right) \Big|_{-1}^{2}$$

$$= 45\pi$$

Example C:

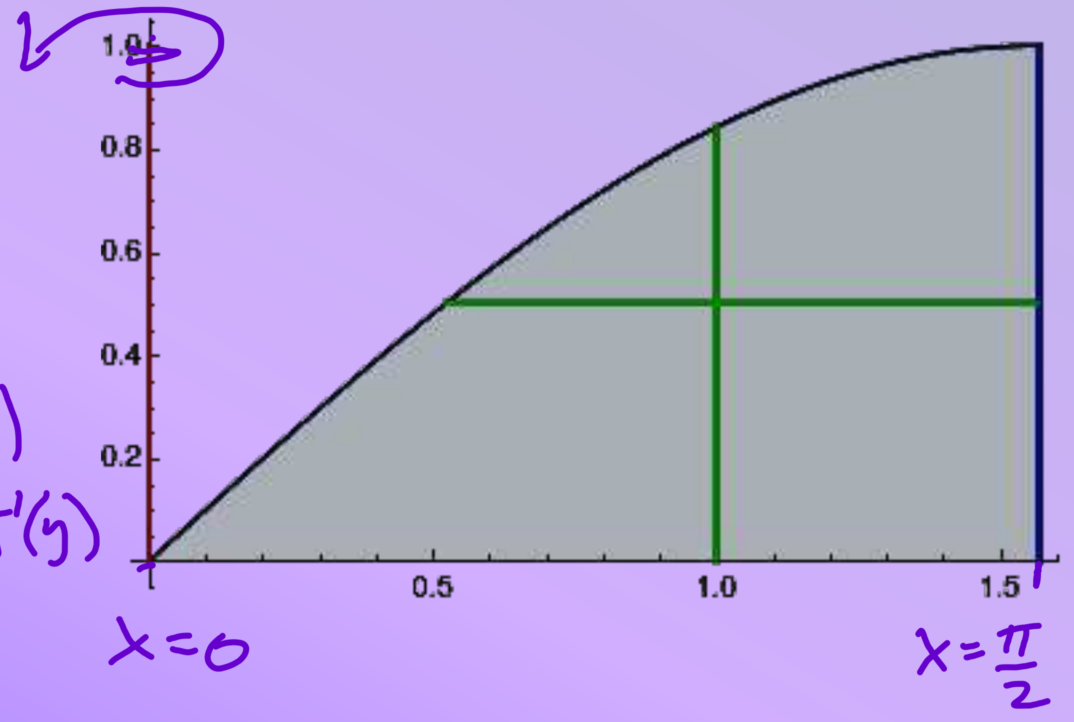
Find the volume of the solid generated by revolving the region bounded by the curve

$$y = \sin(x)$$ (in black) $\rightarrow x = g(y)$

$= \sin^{-1}(y)$

and the x-axis and the lines $x = 0, \frac{\pi}{2}$

about the y-axis.



SHELL METHOD SETUP (Vertical Slices):

$$V = 2\pi \times \int_0^{\frac{\pi}{2}} x\sin(x)dx$$

WASHER METHOD SETUP (Horizontal Slices):

$$V = \pi \times \int_0^1 \left[\frac{\pi^2}{4} - \left(\sin^{-1}(y)\right)^2\right] d\cancel{x}\,y \quad (\ast)$$

→ So to find the antiderivative in (*), the hard part is computing

$$\int_0^1 (\sin^{-1}(y))^2 \, dy$$

→ apply IBP twice, and use a sub. in the second step.

(solution to check your work on the next two slides — we will go over this again on Wednesday)

Notes on how to compute $I = \int_0^1 (\sin^{-1}(y))^2\, dy$.

→ IBP once:

$$\left(\begin{array}{ll} u = \sin^{-1}(y)^2 & dv = dy \\ du = \dfrac{2\sin^{-1}(y)}{\sqrt{1-y^2}}\, dy & v = y \end{array} \right)$$

$$I = y\sin^{-1}(y)^2 \Big|_0^1 - 2\int_0^1 \frac{y}{\sqrt{1-y^2}} \sin^{-1}(y)\, dy$$

(1) $\displaystyle \int \frac{y}{\sqrt{1-y^2}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1-y^2} + C$

$u = 1-y^2$

(2) IBP: $u = \sin^{-1}(y)$ $\quad dv = \dfrac{y\,dy}{\sqrt{1-y^2}}$

$du = \dfrac{dy}{\sqrt{1-y^2}}$ $\quad v = -\sqrt{1-y^2}$

$$I = \left(\frac{\pi}{2}\right)^2 + \underbrace{2\sin^{-1}(y)\sqrt{1-y^2}\Big|_0^1}_{=0} - 2\int_0^1 dy$$

$$= \frac{\pi^2}{4} - 2$$

So $$V = \pi\int_0^1 \left[\frac{\pi^2}{4} - \left(\sin^{-1}(y)\right)^2\right] dy$$

$$= \pi\left[\frac{\pi^2}{4} - \left(\frac{\pi^2}{4} - 2\right)\right] = 2\pi$$